Warsaw U Contest Editorial

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by Mikhail Tikhomirov (MIPT)

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A. Arkanoid

There are k 1×1 square obstacles on a rectangular $n \times m$ field. A ball starts moving diagonally to axes from a certain point. The ball reflects from the walls. When the ball collides with an obstacle, the latter is destroyed and the ball reflects naturally. Determine the time when the last obstacle is destroyed.

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Outline: modelling with effective finding of the next obstacle to break.

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The hardest part is to find for a certain ball position and a set of obstacles (some of the initial ones could get destroyed) which obstacle is the next to be destroyed.

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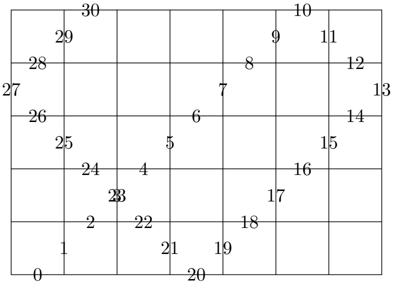
Let us number all possible pairs of position and direction of the ball starting from a certain point.



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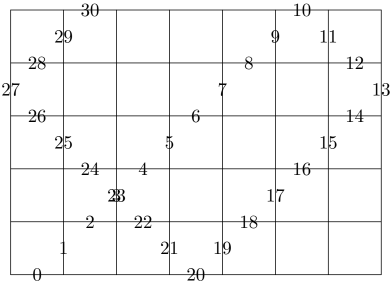

If one has the number k of the situation (position + direction), one can transform it to actual position and velocity coordinates. Indeed, consider time moment k. If the ball started from $(0.5, 0)$ moving to the upper right, it must have made $\lfloor (0.5k + 0.5)/m \rfloor$ bounces from vertical sides and $\lfloor 0.5k/n \rfloor$ from horizontal sides. This information is enough to find the coordinates and velocity.

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To answer a converse question, that is, determine the number k from coordinates and velocity, first notice that the pair (x-coordinate, x-velocity) periodically repeats with period $4m$; the same holds for y and period $4n$. Thus, one has to solve a system of modular equations of sort:

$$k \equiv k_x \pmod{4m}, k \equiv k_y \pmod{4n},$$

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One way of solving a system of similar form is to use the Chinese remainder theorem.

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Now consider a particular obstacle. There are 8 different pairs (position, velocity) that correspond to hitting the obstacle from different angles. Each of these pairs has a unique number (if we use the situations indexing described above). We will store all these numbers in a data structure that supports finding lower/upper bound.

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The solution has $O(k \log k)$ complexity.

B. Vari-directional Streets

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Outline: condense the strongly connected components of the digraph, obtain a simple criterion for a DAG using topsort properties.

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DAG case

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A vertex v is good iff all vertices $u > v$ are reachable from v, and v is reachable from any $u < v$. We will check the first condition for all vertices; the second one can be checked in a completely symmetrical way.

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For a vertex $u > v$ let $deg_{+}(v|u)$ denote the number of edges (w, u) with $w \geqslant v$. Let us call a vertex u a *v-source* if $deg_{+}(v|u) = 0$ (that is, u is a source in the part of the graph to the right of v including v).

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To check the symmetrical condition consider the reversed graph. A vertex v is good if doesn't have a v-source in both cases.

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Both condensation construction and DAG case are solvable in $O(n+m)$ time.

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For details read the enclosed solution.

D. Necklace

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Outline: a Dijkstra-like approach with enough optimizations.

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Let us generate all subsequences in order. We start from the empty sequence. At each moment we will have a data structure of all candidates to be the next minimum. A general idea is to extract minimum from the structure and try all options to expand it by an element it doesn't contain yet.

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Optimization 1: if the size of the structure is greater than k, we can remove the largest entry.

Optimization 2: sort the given numbers (don't forget to store their original indices), stop trying to append a number when the sum becomes too great to fit in the structure.

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While processing the next state we have to try to add number i and add the subset to the list of generated sequences. However, in the sequel we may opt to skip the number. The two transitions are to continue with the number a_i included or not included in the subset; in both cases the current position is increased by 1. Note that the lower bound for the sum does not decrease in any case.

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It is easy to see that the target sequence will not have more than $\log_2 k$ elements since all the subsets of a subsequence precede it in the order. Thus the described solution has complexity $O(n \log n + k \log^2 k)$, since we compare two states in $O(\log_2 k)$ time.

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Outline: if the condition holds, the biconnected blocks have very special form.

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The set S has less than $2l$ edges, and can be decomposed into simple cycles. By assumption, S must itself be a cycle of length l. If that is the case, the intersection of C_1 and C_2 must be a path of length $l/2$. Thus $C_1 \cup C_2$ is a graph that consists of three edge-disjoint paths of length $l/2$ between a pair of vertices v and u.

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Note that no additional edges can be added between vertices of $C_1 \cup C_2$ so that each cycle has length l.

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If $C_1 \cup C_2$ is not the whole component yet, there must be a cycle that has common edges with C_1 or C_2, but doesn't lie completely inside $C_1 \cup C_2$. By a similar argument, the new cycle must consist of two paths of length $l/2$ between v and u: one old and one new.

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Finally, we can decompose the graph into biconnected components in $O(n+m)$ time. We should also check that the cycle lengths for different components are equal.

F. Nim with a twist

Count the number of ways to remove kd Nim heaps out of n so that the second player wins. $d \leqslant 10$, total size of the heaps $\leqslant 10^7$.

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Outline: standard DP with optimization.

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An $O(nd \max a_i)$ solution: $dp_{i,m,x}$ = number of subsets among first i heaps such that the number of omitted heaps is m modulo d, and XOR of all taken heaps' sizes if x.

This DP has $O(nd \max a_i)$ states and transitions.

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Optimization: let us process a_i by increasing. By the time we process a_i, we can't get XOR of some smaller numbers greater than $2a_i$, so we won't store such values. Now appending a single number a_i is done in $O(a_i)$, for the total complexity $O(d\sum a_i + n\log n)$.

G. Parade

In a given tree find a simple path such that the number of edges with exactly one endpoint inside the path is maximized.

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Outline: standard subtree DP.

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First we solve a “vertical” version of the problem. Make the tree rooted, and let *down*(v) be the answer if the path goes from v into its subtree, and we only consider the edges in the subtree (no edge from v to the parent). *down*(v) will allow for a single-vertex path (unlike the original problem).

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If ch_v is the number of children of v, then

$$down(v) = \max\left(ch(v), ch(v) - 1 + \max_{u \text{ is a child of } v} down(u)\right).$$

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if v is a proper LCA of endpoints.

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All the above can be computed in $O(n)$ time.

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Outline: count the standard DP $paths_{v,u,l}$ for the number of all paths without any constraints, then carefully subtract all excess paths.

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We will have to count several DP's. First is the standard $paths_{v,u,l}$ for the total number of paths of length l from v to u:

$$paths_{v,v,0} = 1$$

$$paths_{v,u,l} = \sum_{w:(v,w)\text{ — an edge}} paths_{w,u,l-1}$$

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$paths'_{v,u,l}$ — the number of paths of length l from v to u that contain v only as the start. We want to subtract all non-suitable paths from total. Let l' be the last moment a bad path passes through v. Hence the formula:

$$paths'_{v,u,l} = paths_{v,u,l} - \sum_{l'=1}^{l-1} paths_{v,v,l'} paths'_{v,u,l-l'}$$

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$cycle'_{v,u,l}$ — the number of paths of length l from v to v avoiding u. Let l' be the last moment a bad path passes through u. Then

$$cycle'_{v,u,l} = dp_{v,v,l} - \sum_{l'=1}^{l-1} dp_{v,u,l'}\, paths'_{u,v,l-l'}$$

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Finally, $paths''_{v,u,l}$ is the answer to the original problem. In a similar way we have

$$paths''_{v,u,l} = paths'_{v,u,l} - \sum_{l'=1}^{l-1} paths'_{v,u,l'} \, cycle'_{u,v,l-l'}$$

H. Messenger

$cycle'_{v,u,l}$ — the number of paths of length l from v to v avoiding u. Let l' be the last moment a bad path passes through u. Then

$$cycle'_{v,u,l} = dp_{v,v,l} - \sum_{l'=1}^{l-1} dp_{v,u,l'} paths'_{u,v,l-l'}$$

Finally, $paths''_{v,u,l}$ is the answer to the original problem. In a similar way we have

$$paths''_{v,u,l} = paths'_{v,u,l} - \sum_{l'=1}^{l-1} paths'_{v,u,l'} cycle'_{u,v,l-l'}$$

All the above values can be computed in $O(nml + n^2 l^2)$ time, and each query can be answered in $O(1)$ time.

I. Diligent Johny

We are given a permutation. We repeatedly go from the current permutation to lexicographically previous one using minimal number of element swaps (not necessarily adjacent!). How many swaps we will make in total before we arrive to the $(1, \ldots, n)$ permutation?

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Outline: combinatorial argument, then "number-by-permutation"-like algorithm with RSQ.

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Note that we can just as well go from $(1, \ldots, n)$ to the next permutation until we arrive at the given one. In the sequel we will use this restatement.

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To do that, first we have to get from $(1, \ldots, n)$ to $(1, n, \ldots, 2)$. Next we have to change $(1, n, \ldots, 2)$ into $(2, 1, 3, \ldots, n)$, then to $(2, n, \ldots, 3, 1)$, and so on.

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In general, we have n steps of "swap the suffix" sort. Each of them take $f(n-1)$ steps.

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In general, we have n steps of "swap the suffix" sort. Each of them take $f(n-1)$ steps.

Between these steps we have to change $(k, n, \dots, k+1, k-1, \dots, 1)$ to $(k+1, 1, \dots, k, k+2, \dots, n)$, where $k = 1, \dots, n-1$.

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The minimal number of swaps to change a permutation p into permutation q is equal to n - (number of cycles in $p^{-1}q$).

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One can check with some case analysis (or with brute-force for small numbers) that the number of swaps needed to change $(k, n, \ldots, k+1, k-1, \ldots, 1)$ into $(k+1, 1, \ldots, k, k+2, \ldots, n)$ doesn't depend on k and is equal to $\lceil n/2 \rceil$.

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Thus, we have $f(n) = nf(n-1) + (n-1)\lceil n/2 \rceil$. Values of this recurrence can readily be found in $O(n)$ time.

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By previous arguments, we have to perform $(x_l - 1)(f(n - l) + \lceil (n - l)/2 \rceil)$ swaps. After that, p_l is in its place, and all the later elements are sorted, thus we reduce to a smaller problem.

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Values of x_l can be found using any kind of RSQ data structure in $O(n \log n)$.

J. Not Nim

Two players are playing a game with n pairs of heaps of stones. Initially both heaps in i-th pair contain a_i stones. The first player can remove any number of stones from any heap. The second player must move several stones between heaps in some pair. The first player wants to remove all stones in minimal number of moves, while the second player wants to play as long as possible. Find out the number of moves in the game if both play optimally.

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Outline: evil problem with hard-to-identify cases.

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Couple of simple observations:

- The first player always empties one of the heaps.

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- The first player always empties one of the heaps.
- After the first player emptied a heap, the second player should try to even out the heaps in this pair when this is possible.

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The first player can easily win in $F = \sum_{i=1}^{n}(2 + \lfloor \log_2 a_i \rfloor)$ (we only count first player's moves here), but sometimes he can win faster.

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We will say that the first player can *snatch* a move if he forces the second player to move in a situation when each pair of heaps is either empty or has $(2^k, 2^k)$ stones for certain integer k (probably different for different pairs).

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These are the only situations that help the first player to get under the upper bound F. Indeed, if the second player could skip moves, then F would be the exact number of moves. Thus, the only way to do better than F is to force the second player to do harmful moves.

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Note that if some pair contains unequal heaps, than the second player can effectively skip a move, thus such situations are not appealing to the first player.

J. Not Nim

Suppose that the first player will move in a pair that currently contains (x, x) stones. Forcing a second player's move here will result in $(x-1, 0)$ instead of $(x, 0)$ (if the second player could skip). The move will be harmful if $\lfloor \log_2(x-1) \rfloor < \lfloor \log_2 x \rfloor$, or, equivalently, $x = 2^k$.

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Forcing of a move happens when the first player empties the last heap in some pair. When this happens, we want all other pairs to contain equal heaps (otherwise, the second player can effectively skip).

We want to make sizes of all pairs as close to $(1, 1)$ as possible without making unequal pairs. By making enough moves, a pair (x, x) can be forced to a pair $(2^k - 1, 2^k - 1)$, where k is the number of leading ones in binary representation of x. Moving further in this pair will result in unequal heaps. We suppose that at all times the heaps are reduced to this form.

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Suppose that the second player moves in a $(2^k - 1, 2^k - 1)$ pair. If $k = 0$, then we empty the pair and the forcing continues. Otherwise, after dividing by two the heap turns into $(2^{k-1} - 1, 2^{k-1} - 1)$.

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Both the first and the second player will choose a pair with maximal k.

Indeed, the first player will want the second player to move in small pairs, so he'll eliminate the large ones.

Similarly, in the end the second player wants to have as few $(1, 1)$ pairs as possible, so he'll avoid making them from $(3, 3)$. Similarly, to avoid making $(3, 3)$ he'll avoid moving in $(7, 7)$, and so on.

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Finally, we can model the game as follows. After reducing all heaps to $(2^k - 1, 2^k - 1)$ form we'll store the number of pairs with $k = 0, 1, \ldots, \log_2 A$ (where A is the maximal value of a heap size).

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If you have trouble understanding this solution, try to work out why the answer for $3, 3, 3$ input is 15 instead of 17.

K. Stutter

Find longest common subsequence of two sequences a and b that consists of pairs of equal numbers. You can perform $O(nm)$ operations, but can't store $\Omega(nm)$ memory.

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Outline: optimize the memory with additional bookkeeping.

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The standard DP solution for LCS can be modified as follows. Let $dp_{i,j}$ be equal to the maximal length of LCS of first i and j elements of a and b respectively if we are forced to take both elements of each pair simultaneously.

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Let $prev(a, i, c)$ be the last occurence of c in a before position i. Then $dp_{i,j} = \max(dp_{i-1,j}, dp_{i,j-1})$ if $a_i \neq b_j$, and $\max(dp_{i-1,j}, dp_{i,j-1}, 2 + dp_{prev(a,i,c)-1,prev(b,j,c)-1})$ if $a_i = b_j = c$ (all $prev$'s have to be defined, of course).

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Note that we can use $dp_{prev(b,j,c)}^{equal}$ in place of $dp_{prev(a,i,c)-1,prev(b,j,c)-1}$. This eliminates our need for $\Omega(nm)$ memory and requires only $O(n+m)$ memory.